TORSION-FREE COVERS

BY

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ABSTRACT

This paper studies the existence and properties of a torsion-free cover with respect to a faithful hereditary torsion theory $(\mathscr{T}, \mathscr{F})$ of modules over a ring with unity. A direct sum of a finite number of torsion-free covers of modules is the torsion-free cover of the direct sum of the modules. The concept of a $\mathscr{T}$-neat homomorphism, which generalizes Enochs' definition of a neat submodule, is introduced and studied. This allows the generalization of a result of Enochs on liftings of homomorphisms. Hereditary torsion theories for which every module has a torsion-free cover are called universally covering. If the inclusion map of R into the appropriate quotient ring Q is a left localization in the sense of Silver, the problem of the existence of universally-covering torsion theories can be reduced to the case $R = Q$. As a consequence, many sufficient conditions for a hereditary torsion theory to be universally covering are obtained. For a universally-covering hereditary torsion theory $(\mathscr{T}, \mathscr{F})$, the following conditions are equivalent: (1) the product of $\mathscr{T}$-neat homomorphisms is always $\mathscr{T}$-neat; (2) the product of torsion-free covers is always $\mathscr{T}$-neat; (3) every nonzero module in $\mathscr{T}$ has a nonzero socle.

0. Introduction

The concept of a torsion-free cover for the usual torsion theory over an integral domain was introduced by Enochs [3, 4]. The definition of torsion-free cover was extended to perfect torsion theories by Banaschewski [1] and to faithful hereditary torsion theories by Teply [11].

In this paper we continue the study of torsion-free covers with respect to faithful hereditary torsion theories of modules over a ring with unity. The first section summarizes our notation and terminology.

In Section 2, we examine the properties of a torsion-free cover. For example, the direct sum of the torsion-free covers of modules M_i $(i = 1, 2, \cdots, n)$ is the torsion-free cover of $\oplus_{i=1}^{n} M_i$. We also introduce the concept of relative neatness which allows us to generalize the main result of [4] on liftings of homomorphisms.

In Section 3, we consider those torsion theories for which every module has a torsion-free cover. Such torsion theories are called universally covering. If the

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inclusion of R into the quotient ring Q of the theory is a left localization in the sense of Silver [9], then the problem reduces to the case $R = Q$ (Theorem 3.4.) This allows us to deduce as corollaries that perfect torsion theories and torsion theories with noetherian quotient rings are universally covering. If the quotient ring for a primary torsion theory in the sense of Dickson [2] is a proper left localization, then that theory is universally covering. If the localization functor of a torsion theory is exact, then the theory is universally covering if and only if it is perfect (Theorem 3.13). In particular, if R is left hereditary, then a torsion theory is perfect if and only if it is universally covering; and if R is left semi-hereditary, then a sufficient condition for a torsion theory to be universally covering is that its associated filter have a cofinal subset of finitely generated left ideals.

The final section studies products of torsion-free covers for universally covering torsion theories.

1. Preliminaries

Throughout the following, R will denote an associative ring with unit element 1 and all modules and morphisms will be taken from the category R-mod of unitary left R-modules unless the contrary is specifically indicated. Morphisms will always be written as acting on the side opposite scalar multiplication. The injective hull of a module will be denoted by $E(M)$.

A torsion theory on R-mod, which was first introduced by Dickson [2], consists of a pair $(\mathscr{T}, \mathscr{F})$ of classes of modules satisfying the conditions

$$\mathscr{T} = \{{}_RT \mid \mathrm{Hom}_R(T,F) = 0 \text{ for all } F \in \mathscr{F}\}$$

and

$$\mathscr{F} = \{{}_RF \mid \mathrm{Hom}_R(T,F) = 0 \text{ for all } T \in \mathscr{T}\}.$$

The modules in $\mathscr{T}$ are called *torsion*; the modules in $\mathscr{F}$ are called *torsion-free*. A submodule N of a module M is called *$\mathscr{T}$-pure* in M if $M/N \in \mathscr{F}$. Every module M has a unique maximal $\mathscr{T}$-pure torsion submodule $\mathscr{T}(M)$, called *the torsion submodule* of M.

A torsion theory $(\mathscr{T},\mathscr{F})$ is called *hereditary* if $\mathscr{T}$ is closed under submodules. Given a hereditary torsion theory $(\mathscr{T},\mathscr{F})$, a submodule N of M is called *$\mathscr{T}$-dense* in M if $M/N \in \mathscr{T}$. The set $\mathscr{L}$ of $\mathscr{T}$-dense left ideals of R forms an idempotent filter [5]. If $\mathrm{Hom}_R(M, V) \to \mathrm{Hom}_R(N, V) \to 0$ is exact for every $\mathscr{T}$-dense submodule N of M, then the module V is called *$\mathscr{T}$-injective*. For each module M, there is a smallest $\mathscr{T}$-injective submodule $E_{\mathscr{T}}(M)$ of $E(M)$, which is called the *$\mathscr{T}$-injective hull* of M; $E_{\mathscr{T}}(M)$ is unique up to isomorphism [8]. Let $\mathscr{E}$ be the class of all $\mathscr{T}$-injective modules in $\mathscr{F}$; then $\mathscr{E}$ is precisely the class of all members of $\mathscr{F}$ which are $\mathscr{T}$-pure in every torsion-free module containing them. For a module M, the *quotient module* of M is $Q_{\mathscr{T}}(M) = E_{\mathscr{T}}(M/\mathscr{T}(M))$. The endomorphism ring of $Q_{\mathscr{T}}(R)$ is called the *quotient ring* of R with respect to $(\mathscr{T}, \mathscr{F})$ and is denoted by $Q_{\mathscr{T}}$. If $R \in \mathscr{F}$, then $(\mathscr{T}, \mathscr{F})$ is called *faithful*. For faithful hereditary torsion theories, R can be canonically embedded in $Q_{\mathscr{T}}$, and $Q_{\mathscr{T}} \cong Q_{\mathscr{T}}(R)$ as left R-modules. In addition, if $M \in \mathscr{F}$, then $E_{\mathscr{T}}(M)$ is canonically a left $Q_{\mathscr{T}}$-module. The classes $\mathscr{T}$, $\mathscr{F}$, $\mathscr{L}$, and $\mathscr{E}$ uniquely determine each other; therefore, to specify the hereditary torsion theory, it suffices to determine any one of these four classes.

Unless stated otherwise, *in the remainder of this paper* $(\mathscr{T}, \mathscr{F})$ *will always denote a faithful hereditary torsion theory on* R-mod, *and* $\mathscr{E}$ *and* $\mathscr{L}$ *will always denote the classes defined in the preceding paragraph.* For the basic properties of the above concepts, the reader should consult [2, 5, 6, 8, 10].

A ring homomorphism $R \to S$ is called a *left localization* [9] if it is an epimorphism in the category of rings and S_R is flat. If $S \neq R$, then the left localization is said to be *proper*.

PROPOSITION 1.1. *The following conditions are equivalent for* $(\mathscr{T}, \mathscr{F})$.

(1) *If* $M \in \mathscr{F}$, *then* $Q_{\mathscr{T}} \otimes_R M \in \mathscr{F}$.

(2) *If* $M \in \mathscr{F}$, $\xi_M : Q_{\mathscr{T}} \otimes M \to Q_{\mathscr{T}}(M) : \Sigma(q_i \otimes m_i) \mapsto \Sigma q_i m_i$ *is a monomorphism.*

(3) *The ring inclusion* $R \to Q_{\mathscr{T}}$ *is a left localization.*

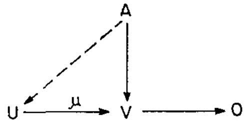
PROOF. By straightforward computation, the reader can verify that $\ker(\xi_M) = \mathscr{T}(Q_{\mathscr{T}} \otimes_R M)$, and consequently (1)$\Leftrightarrow$(2).

From (2) it follows that $\xi_{Q_{\mathscr{T}}} : Q_{\mathscr{T}} \otimes_R Q_{\mathscr{T}} \to Q_{\mathscr{T}}$ is an isomorphism and hence $R \to Q_{\mathscr{T}}$ is an epimorphism in the category of rings by [9, Prop. 1.1]. Since each left ideal I is torsion free, (2) implies that $Q_{\mathscr{T}} \otimes_R I \to Q_{\mathscr{T}} \otimes_R R \cong Q_{\mathscr{T}}$ is a monomorphism. Thus $Q_{\mathscr{T}}$ is flat as a right R-module [7, p. 132]; so (2) $\Rightarrow$ (3).

Finally, (3) $\Rightarrow$ (1) is easily deduced from [9, Prop. 1.7].

2. Torsion-free covers

If $\mathscr{A}$ is a class of left R-modules, then we denote by $\pi^{-1}(\mathscr{A})$ the class of all epimorphisms $\mu : {}_RU \to {}_RV$ such that for any member A of $\mathscr{A}$, any diagram of the form of Fig 1. can be completed commutatively. An epimorphism $\mu : F \to M$ is called an $\mathscr{F}$*-precover* provided that $F \in \mathscr{F}$ and that $\mu \in \pi^{-1}(\mathscr{F})$. An $\mathscr{F}$-precover is called an $\mathscr{F}$*-cover* provided that no nonzero $\mathscr{T}$-pure submodules of F are



Fig, 1

contained in $\ker(\mu)$. The concept of a torsion-free cover has been studied for the special case of integral domains in [3, 4] and for abstract torsion theories $(\mathscr{T},\mathscr{F})$ in [11]. If the $\mathscr{F}$-cover of a module M exists, it is known to be unique up to isomorphism (see proof of [11, Theorem 2.4]), and we denote it by $\mathscr{F}(M)$.

The next lemma shows why, in studying torsion-free covers, it is reasonable to limit ourselves to faithful torsion theories. For the following lemma only, we will drop our standing assumption that $(\mathscr{T},\mathscr{F})$ is faithful.

LEMMA 2.1. *If $(\mathscr{T},\mathscr{F})$ is an arbitrary torsion theory and R has an $\mathscr{F}$-precover, then every module cogenerated by R is torsion free. In particular, every projective module is torsion free.*

PROOF. Let $\mu: F\to R$ be an $\mathscr{F}$-precover. For $x\in F$, $\mathscr{T}(R)x\in\mathscr{T}\cap\mathscr{F}=0$. Therefore $\mathscr{T}(R)F=0$, and so $0=\mathscr{T}(R)F\mu=\mathscr{T}(R)R=\mathscr{T}(R)$. Therefore $R\in\mathscr{F}$. Since $\mathscr{F}$ is closed under submodules and direct products, any module cogenerated by R is also in $\mathscr{F}$.

Except for [4, Cor. 2, p. 43], all of the results proved by Enochs in the first two sections of [4] for the special case of the standard torsion theory over an integral domain generalize directly to abstract torsion theories.

The following theorem shows that, in some sense, the property of being an $\mathscr{F}$-cover is "local".

THEOREM 2.2. *Let $\mu: F\to M$ be an epimorphism from a torsion-free module F onto a module M, and let M be a directed union of a set of its submodules $\{N_i\}$. If the restriction μ_i of μ to $N_i\mu^{-1}$ is an $\mathscr{F}$-cover for each i, then μ is an $\mathscr{F}$-cover.*

PROOF. For each i let $F_i=N_i\mu^{-1}$, and let $\sigma_i: F_i\to F$ be the inclusion map. Then F is the directed union of the F_i. If $G\in\mathscr{F}$ and if $\alpha\in\operatorname{Hom}(G,M)$, set $G_i=N_i\alpha^{-1}$ for each i. Since $\mu_i\in\pi^{-1}(\mathscr{F})$, there exists $\beta_i\in\operatorname{Hom}(G_i,F_i)$ such that $\beta_i\mu_i=\alpha|_{G_i}$. Since G is the directed union of the G_i, there is a unique homomorphism $\beta: G\to F$, the restriction of which to any G_i is $\beta_i\sigma_i$. Thus $\beta\mu=\alpha$, and so $\mu\in\pi^{-1}(\mathscr{F})$.

If H is a $\mathscr{T}$-pure submodule of F contained in $\ker(\mu)$, then $H\sigma_i^{-1}\subseteq\ker(\mu_i)$ for each i. Each σ_i induces a monomorphism $F_i/H\sigma_i^{-1} \to F/H$. Therefore $F_i/H\sigma_i^{-1} \in \mathscr{F}$, which forces $H\sigma_i^{-1} = 0$; thus $H = 0$. Hence $\mu: F \to M$ is an $\mathscr{F}$-cover.

COROLLARY 2.3. *An epimorphism $\mu: F \to M$ from a torsion-free module F to a module M is an $\mathscr{F}$-cover if the restriction of μ to the inverse image of any finitely-generated submodule of M is an $\mathscr{F}$-cover.*

PROPOSITION 2.4. *Let M be a module having an $\mathscr{F}$-cover. Then M is $\mathscr{T}$-injective [injective] if and only if $\mathscr{F}(M)$ is $\mathscr{T}$-injective [injective].*

PROOF. We will only prove the case of $\mathscr{T}$-injectivity, since the case of injectivity is similar. Assume that M is $\mathscr{T}$-injective. Since $E_{\mathscr{T}}(\mathscr{F}(M))/\mathscr{F}(M) \in \mathscr{T}$, the $\mathscr{T}$-injectivity of M implies the existence of a homomorphism α making the diagram (Fig. 2) commute (λ canonical). Since $\mathscr{F}(M) \in \mathscr{F}$, so is $E_{\mathscr{T}}(\mathscr{F}(M))$; thus by the

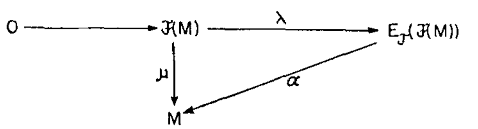


Fig. 2

definition of an $\mathscr{F}$-cover, there is a homomorphism $\beta: E_{\mathscr{T}}(\mathscr{F}(M)) \to \mathscr{F}(M)$ satisfying $\beta\mu = \alpha$. By a generalization of [3, Th. 2] (see [11, §2]), $\lambda\beta$ is an automorphism of $\mathscr{F}(M)$, which proves that $\mathscr{F}(M)$ is a direct summand of $E_{\mathscr{T}}(\mathscr{F}(M))$. Hence $\mathscr{F}(M) \in \mathscr{E}$.

Conversely, assume that $\mathscr{F}(M)$ is $\mathscr{T}$-injective. To prove that M is $\mathscr{T}$-injective, it suffices to show that every homomorphism $I \to M$ can be extended to a homomorphism $R \to M$, where $I \in \mathscr{L}$ [8, Prop. 0.5]. Since $R \in \mathscr{F}$, every I in $\mathscr{L}$ is in $\mathscr{F}$; so for any $\alpha: I \to M$, there exists a $\beta: I \to \mathscr{F}(M)$ with $\alpha = \beta\mu$. Since $\mathscr{F}(M) \in \mathscr{E}$, there then exists a $\gamma: R \to \mathscr{F}(M)$ extending β, and thus $\gamma\mu: R \to M$ extends α.

LEMMA 2.5. *Let $\mu: F \to M$ be an $\mathscr{F}$-cover, and let N be a submodule of M such that $N\mu^{-1}$ is $\mathscr{T}$-pure in F. Then the restriction of μ to $N\mu^{-1}$ is an $\mathscr{F}$-cover of N.*

PROOF. Let $G = N\mu^{-1}$ and let μ' be the restriction of μ to G. Clearly μ' is an $\mathscr{F}$-precover. If W is a $\mathscr{T}$-pure submodule of G contained in $\ker(\mu')$, then the exactness of the sequence

$$0 \to G/W \to F/W \to F/G \to 0$$

and the fact that $F/G \in \mathscr{F}$ imply that $F/W \in \mathscr{F}$. Since μ is an $\mathscr{F}$-cover, then $W = 0$. Therefore, μ' is also an $\mathscr{F}$-cover.

COROLLARY 2.6. *If $E(M)$ has an $\mathscr{F}$-cover, so does $E_{\mathscr{T}}(M)$.*

Generalizing a concept introduced in [4], we say that $\alpha \in \mathrm{Hom}(N, M)$ is *$\mathscr{T}$-neat* if and only if, for each $\mathscr{T}$-dense submodule U of V and each $\sigma \in \mathrm{Hom}(U, N)$, the following conditions are equivalent:

(i) There exists a submodule W of V properly containing U and a $\tau \in \mathrm{Hom}(W, M)$ such that $\tau|_U = \sigma\alpha$.

(ii) There exists a submodule W' of V properly containing U and a $\tau' \in \mathrm{Hom}(W', N)$ such that $\tau'|_U = \sigma$.

A straightforward argument establishes that in order to prove that a homomorphism α is $\mathscr{T}$-neat, it suffices to consider the case $V = R$. We call a submodule N of a module M *$\mathscr{T}$-neat* if and only if the inclusion map $N \to M$ is $\mathscr{T}$-neat. In case R is a commutative integral domain and $(\mathscr{T}, \mathscr{F})$ is the usual torsion theory, our definition of "$\mathscr{T}$-neat" coincides with Enochs' definition of "neat" in [4]; in the special case where R is the ring of integers, our definition of "$\mathscr{T}$-neat" coincides with the usual definition of "neat" for abelian groups. The straightforward generalization of [4, Th. 2.1] shows that every $\mathscr{F}$-cover is a $\mathscr{T}$-neat homomorphism.

The proof of the following lemma is left to the diagram-chasing of the reader:

LEMMA 2.7. *Let α: $M \to N$ and β: $N \to U$ be R-homomorphisms. Then the following statements are valid.*

(1) *If α and β are both $\mathscr{T}$-neat, so is $\alpha\beta$.*

(2) *If $\alpha\beta$ is $\mathscr{T}$-neat, then so is α.*

LEMMA 2.8. *The direct sum of a finite number of $\mathscr{T}$-neat homomorphisms is $\mathscr{T}$-neat.*

PROOF. Let α_i: $U_i \to V_i$ $(i = 1, \cdots, n)$ be $\mathscr{T}$-neat homomorphisms, and set $U = \oplus U_i$, $V = \oplus V_i$, and $\alpha = \oplus \alpha_i$: $U \to V$. For each i, we have the canonical projections π_i: $U \to U_i$ and π_i': $V \to V_i$.

Let $I \in \mathscr{L}$, $\sigma: I \to U$, and assume that there exists a left ideal K of R properly containing I and a homomorphism $\tau: K \to V$, the restriction of which to I is $\sigma\alpha$. Then $\tau\pi_1'$ extends $\sigma\pi_1\alpha_1$. Hence, by the $\mathscr{T}$-neatness of α_1, there exists a left ideal K_1 of R contained in K and properly containing I and a homomorphism $\theta_1: K_1 \to U_1$ extending $\sigma\pi_1$.

The restriction τ_1 of τ to K_1 extends $\sigma\alpha$, and hence $\tau_1\pi_2'$ extends $\sigma\pi_2\alpha_2$. Since α_2 is $\mathscr{T}$-neat, there then exists a left ideal K_2 of R contained in K_1 and properly containing I and a homomorphism $\theta_2\colon K_2 \to U_2$ extending $\sigma\pi_2$.

Continuing in this manner, we obtain left ideals

$$K_1 \supseteq K_2 \supseteq \cdots \supseteq K_n \supsetneq I$$

and homomorphisms $\theta_i\colon K_i \to U_i$ $(i = 1,\cdots,n)$ such that each θ_i extends $\alpha\pi_i$. Define $\theta\colon K_n \to U$ by $\theta\colon a \mapsto (a\theta_1, \cdots, a\theta_n)$. For $a \in I$, $a\theta = (a\theta_1, \cdots, a\theta_n) = (a\sigma\pi_1,\cdots,a\sigma\pi_n) = a\sigma$; so θ is a proper extension of σ. This proves that α is $\mathscr{T}$-neat.

Proposition 2.9. *A submodule of a torsion-free module is $\mathscr{T}$-neat if and only if it is $\mathscr{T}$-pure.*

Proof. Let $M \in \mathscr{F}$, and let N be a $\mathscr{T}$-neat submodule of M. Assume further that $W/N = \mathscr{T}(M/N) \neq 0$. If $\lambda\colon N \to W$ is the inclusion map, then by $\mathscr{T}$-neatness there is a submodule W' of W properly containing N and a homomorphism $\beta\colon W' \to N$ such that $\lambda\beta$ is the identity map on N. Thus N is a direct summand of W', which contradicts the fact that $W'/N \in \mathscr{T}$ and $W' \in \mathscr{F}$.

Conversely, let $M \in \mathscr{F}$, and let N be $\mathscr{T}$-pure in M. Let $V/U \in \mathscr{T}$ and let $\alpha \in \mathrm{Hom}\,(U,N)$. Assume that there exists a submodule W of V properly containing U and a homomorphism $\beta\colon W \to M$, the restriction of which to U equals α. Then β induces a homomorphism $\beta'\colon W/U \to M/N$ given by $(w + U)\beta' = w\beta + N$. Since $W/U \in \mathscr{T}$ and $M/N \in \mathscr{F}$, β' must equal 0, which implies that $W\beta \subseteq N$. Hence N is $\mathscr{T}$-neat in M.

Proposition 2.10. *If $F \in \mathscr{F}$ and if $\phi\colon F \to M$ is a $\mathscr{T}$-neat homomorphism, then any $\mathscr{T}$-pure submodule of F contained in* $\ker(\phi)$ *is in $\mathscr{E}$.*

Proof. Let W be a $\mathscr{T}$-pure submodule of F contained in $\ker(\phi)$. If $I \in \mathscr{L}$ and $\alpha \in \mathrm{Hom}\,(I, W)$, a standard argument yields a maximal extension $\alpha'\colon I' \to W$ of α, where $I \subseteq I' \subseteq R$. By [6, Prop. 3.2], it is sufficient to prove that $I' = R$.

If $I' \neq R$, then the 0-homomorphism $R \to M$ properly extends $\alpha'\phi$. By the $\mathscr{T}$-neatness of ϕ, there exists a left ideal H of R properly containing I' and a $\beta \in \mathrm{Hom}(H,F)$ extending α'. By the maximality of I', $H\beta \nsubseteq W$; so $0 \neq [H\beta + W]/W \subseteq F/W$. Since W is $\mathscr{T}$-pure in F, then $[H\beta + W]/W \in \mathscr{F}$. But by the maximality of I',

$$[H\beta + W]/W \cong H\beta/[H\beta \cap W] = H\beta/I'\beta \cong H/I' \in \mathscr{T},$$

and hence $[H\beta + W]/W = 0$. Thus $H\beta \subseteq W$, which is a contradiction.

THEOREM 2.11. *The direct sum of the $\mathscr{F}$-covers of a finite number of modules is the $\mathscr{F}$-cover of their direct sum.*

PROOF. Let $M_1, \cdots, M_n$ be modules having $\mathscr{F}$-covers μ_i: $F_i \to M_i$. Define $M = \oplus M_i$, $F = \oplus F_i$, and $\mu = \oplus \mu_i$: $F \to M$. Then μ is clearly an $\mathscr{F}$-precover. By Lemma 2.8, μ is $\mathscr{T}$-neat; so by Proposition 2.10, any $\mathscr{T}$-pure submodule contained in $\ker(\mu)$ is in $\mathscr{E}$. Hence it is sufficient to prove by induction on n that any submodule of $\ker(\mu)$ in $\mathscr{E}$ must equal zero. From the definition of an $\mathscr{F}$-cover, this is certainly true for $n = 1$. Assume inductively that there are no nonzero submodules of $\ker(\oplus_{i=1}^{k} \mu_i)$ in $\mathscr{E}$. Let V be a nonzero submodule of $\ker(\oplus_{i=1}^{k+1} \mu_i)$ in $\mathscr{E}$. For each i, let π_i: $V \to F_i$ be the restriction of the canonical projection $F \to F_i$. Then π_{k+1} is not a monomorphism; for otherwise $V\pi_{k+1}$ would be a nonzero submodule of $\ker(\mu_{k+1})$ in $\mathscr{E}$. On the other hand, if π_{k+1} is not a monomorphism, then $\ker(\pi_{k+1}) \subseteq \ker(\oplus_{i=1}^{k} \mu_i)$ and is in $\mathscr{E}$ by [6, Prop. 3.3], which contradicts the induction hypothesis. Therefore $V = 0$.

PROPOSITION 2.12. *Let every $\mathscr{T}$-neat submodule of M have an $\mathscr{F}$-cover, and let μ: $\mathscr{F}(M) \to M$ be an $\mathscr{F}$-cover of M. Then the following statements about a submodule N of M are equivalent*:

(1) *N is a $\mathscr{T}$-neat submodule of M.*

(2) *The restriction of μ to some $\mathscr{T}$-neat submodule of $\mathscr{F}(M)$ is an $\mathscr{F}$-cover of N.*

(3) *There exists a $\mathscr{T}$-neat submodule G of $\mathscr{F}(M)$ such that $N = G\mu$ and $\mu|_G \in \pi^{-1}(\mathscr{L})$.*

PROOF. (1) $\Rightarrow$ (2): If N is a $\mathscr{T}$-neat submodule of M, then the restriction of μ to $N\mu^{-1}$ belongs to $\pi^{-1}(\mathscr{F})$. Using a straightforward modification of [4, Prop. 2.3], the reader can easily verify that there exists a direct summand G of $N\mu^{-1}$ such that the restriction μ'' of μ to G is an $\mathscr{F}$-cover of N. Thus $\lambda'\mu = \mu''\lambda$, where λ': $G \to \mathscr{F}(M)$ and λ: $M \to N$ are the inclusion maps. Since μ'': $G \to N$ is an $\mathscr{F}$-cover, μ'' is $\mathscr{T}$-neat, and λ is $\mathscr{T}$-neat by hypothesis. By Lemma 2.7(1), $\mu''\lambda = \lambda'\mu$ is $\mathscr{T}$-neat, and thus by Lemma 2.7(2), λ' is also $\mathscr{T}$-neat.

(2) ⇒ (3): Take G to be the appropriate covering module for N.

(3) ⇒ (1): Suppose that $G\mu = N$, where G is $\mathscr{T}$-neat in $\mathscr{F}(M)$ and $\mu|_G \in \pi^{-1}(\mathscr{L})$. Let $I \in \mathscr{L}$ and let $\alpha \in \operatorname{Hom}(I, N)$. Suppose that $\beta: H \to M$ extends α, where H is a left ideal of R properly containing I. Then Fig. 3 commutes:

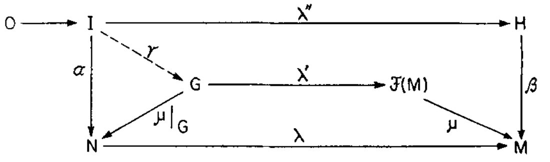


Fig. 3

($\lambda, \lambda', \lambda''$ are the inclusion maps).

Since $\mu|_G \in \pi^{-1}(\mathscr{L})$, there exists $\gamma \in \operatorname{Hom}(I, G)$ satisfying $\alpha = \gamma\mu|_G$. Also, λ' is $\mathscr{T}$-neat by hypothesis, and μ is $\mathscr{T}$-neat because it is an $\mathscr{F}$-cover. By Lemma 2.7, $\lambda'\mu$ is $\mathscr{T}$-neat. Moreover, $\gamma\lambda'\mu = \gamma(\mu|_G)\lambda = \alpha\lambda = \beta|_I$. Since $\lambda'\mu$ is $\mathscr{T}$-neat there exists a left ideal K of R proper.y containing I and a homomorphism $\phi: K \to G$, the restriction of which to I equals γ. Hence $\alpha = \gamma\mu|_G = (\phi|_I)(\gamma|_G)$; so $\phi\mu|_G$ extends α to K. This proves that λ is $\mathscr{T}$-neat.

Using terminology analogous to that of [4], we call a homomorphism β in the commutative diagram below a *lifting* of $\alpha: M \to N$, where M and N are modules having respective $\mathscr{F}$-covers $\mu: \mathscr{F}(M) \to M$ and $\mu': \mathscr{F}(N) \to N$ (Fig. 4):

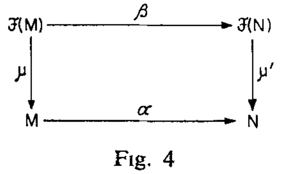


Fig. 4

By definition of μ', such liftings always exist, but they may not be unique.

THEOREM 2.13. *Suppose that every $\mathscr{T}$-neat submodule of M has an $\mathscr{F}$-cover. Let $\mu: \mathscr{F}(M) \to M$ and $\mu': \mathscr{F}(N) \to N$ be $\mathscr{F}$-covers, and let $\beta: \mathscr{F}(M) \to \mathscr{F}(N)$ be a lifting of $\alpha: M \to N$. Then the following statements are equivalent*:

(1) *β is an isomorphism.*

(2) (a) *α is an epimorphism*;

(b) $\operatorname{Ext}^1(G, \ker(\alpha)) = 0$ *for all* $G \in \mathscr{F}$;

(c) $\ker(\alpha)$ *contains no non-zero $\mathscr{T}$-neat submodules of M*;

(d) *If U is a $\mathscr{T}$-pure submodule of $\mathscr{F}(M)$ and if $U\mu \subseteq \ker(\alpha)$, then $\mu|_U \in \pi^{-1}(\mathscr{L})$.*

(3) *Every lifting of* α *is an isomorphism.*

REMARK. If every left ideal in $\mathscr{L}$ is projective, then condition 2(d) of Theorem 2.13 holds; so Theorem 2.13 is a generalization of the main result of [4] in which Enochs proves the result for the usual torsion theory over a Dedekind domain [4, Th. 3.1].

PROOF. (1) $\Rightarrow$ (2): Since $\beta\mu' = \mu\alpha$ and μ' is an epimorphism, α is an epimorphism whenever β is. Thus (1) $\Rightarrow$ (2a).

If H is a $\mathscr{T}$-neat submodule of $\ker(\alpha)$, then by Proposition 2.12 there is a $\mathscr{T}$-neat submodule G of $\mathscr{F}(M)$ such that $G\mu = H$. Thus to prove that (1) implies (2c) and (2d), it is sufficient to prove that any $\mathscr{T}$-neat submodule K of $\mathscr{F}(M)$ with $K\mu \subseteq \ker(\alpha)$ must be zero. By Proposition 2.9, any such K is $\mathscr{T}$-pure in $\mathscr{F}(M)$. Since $\mu\alpha = \beta\mu'$, (1) implies that $K\beta$ is a $\mathscr{T}$-pure submodule of $\mathscr{F}(N)$ contained in $\ker(\mu')$. Hence $K\beta = 0$; so by (1), $K = 0$ as desired.

It remains to show that (1) $\Rightarrow$ (2b). Since μ' is an $\mathscr{F}$-cover and β is an isomorphism, a simple diagram chase shows that $\alpha \in \pi^{-1}(\mathscr{F})$ and hence that, in the following commutative diagram with exact rows, α_* is an epimorphism (Fig. 5).

$$\begin{array}{ccccccccc} & & & & & & \mathrm{Ext}^1(G,\mathscr{F}(M)) & \xrightarrow{\beta_*} & \mathrm{Ext}^1(G,\mathscr{F}(N)) \\ & & & & & & \downarrow & & \downarrow \\ \mathrm{Hom}(G,M) & \xrightarrow{\alpha_*} & \mathrm{Hom}(G,N) & \longrightarrow & \mathrm{Ext}^1(G,\ker(\alpha)) & \longrightarrow & \mathrm{Ext}^1(G,M) & \longrightarrow & \mathrm{Ext}^1(G,N) \end{array}$$

Fig. 5

Since β_* is an isomorphism by (1) and since the vertical maps are isomorphisms by an easy generalization of [4, Corol. 3, p. 41], then $\mathrm{Ext}^1(G,\ker(\alpha)) = 0$ follows from the exactness of the diagram.

(2) $\Rightarrow$ (1): By 2(a) and 2(b), $\alpha \in \pi^{-1}(\mathscr{F})$. Thus by the definition of μ, there exists $\gamma: \mathscr{F}(N) \to \mathscr{F}(M)$ such that $\mu' = \gamma\beta\mu'$. Since μ' is an $\mathscr{F}$-cover, $\gamma\beta$ is an automorphism of $\mathscr{F}(N)$ by a generalization of [3, Th. 2] (see [11, §2]). Thus $\ker(\beta)$ is a summand of $\mathscr{F}(M)$. Also $\mu(\ker(\beta)) \subseteq \ker(\alpha)$; so $\mu|_{\ker(\beta)} \in \pi^{-1}(\mathscr{L})$ by 2(d). By Proposition 2.12, $\ker(\mu)$ must be $\mathscr{T}$-neat in M; so $\ker(\beta) \subseteq \ker(\mu)$ by 2(c). Since μ is an $\mathscr{F}$-cover, it follows that $\ker(\beta) = 0$.

(3) $\Leftrightarrow$ (1): Since condition (2) depends only on α and μ (and not on β), it follows from (1) $\Leftrightarrow$ (2) that if one lifting of α is an isomorphism, then so is every other lifting of α.

EXAMPLE 2.14. Let R be a bounded Dedekind prime ring, let M be a maximal two-sided ideal such that $R/M \in \mathscr{T}$ for some $(\mathscr{T},\mathscr{F})$. Then the natural map

$\alpha\colon R/M^k \to R/M^n$ ($k \geqq n \geqq 1$) is easily seen to satisfy condition (2) of Theorem 2.13. (The existence of the required covers can be justified by Corollary 3.17 below.)

EXAMPLE 2.15. Let R be a 3×3 upper triangular matrix ring over a field. Let $\mathscr{T}$ be the class of singular left R-modules; i.e., let $\mathscr{L}$ be the set of essential left ideals. Set $M = Re_{33}/Re_{13}$ and $N = Re_{33}/Re_{23}$, where e_{ij} denotes the matrix with 1 at the i, j entry and zeros elsewhere. Then the natural homomorphism $\alpha\colon M \to N$ satisfies condition (2) of Theorem 2.13. (Again the existence of the required covers comes from Corollary 3.17 below.)

EXAMPLE 2.16. Let $\mathscr{T}$ be closed under injective envelopes (see [10, §4]), and let M be an indecomposable injective module in $\mathscr{T}$. Then M and 0 are the only $\mathscr{T}$-neat submodules of M. Suppose that $\mathscr{F}(M)$ exists and that all of its $\mathscr{T}$-pure submodules are summands. If $\mathscr{F}(N)$ exists, then a lifting β of $\alpha\colon M \to N$ is an isomorphism if and only if $\alpha \in \pi^{-1}(\mathscr{F})$. (Note that Example 2.15 is a special case of Example 2.16.)

3. Universally-covering torsion theories

A torsion theory $(\mathscr{T}, \mathscr{F})$ on R-mod is called *universally covering* if and only if every left R-module has an $\mathscr{F}$-cover. In [3] Enochs proves that the usual torsion theory on a commutative integral domain is universally covering. To study such theories we begin by introducing a construction based on that given by Banaschewski in [1].

For any module M, define $U(M) = \mathrm{Hom}_R(Q_{\mathscr{T}}, E_{\mathscr{T}}(M))$ and $V(M) = \{\alpha \in U(M) \mid R\alpha \subseteq M\}$. Then $U(M)$ is a left $Q_{\mathscr{T}}$-module with multiplication defined by $(q')q\alpha = (q'q)\alpha$, and $V(M)$ is an R-submodule of $U(M)$. Note that, as R-modules, $V(M)$ is large in $U(M)$. Indeed, if $0 \neq \alpha \in U(M)$, then $(1)\alpha \in E_{\mathscr{T}}(M)$. By the largeness of M in $E_{\mathscr{T}}(M)$, there exists an $r \in R$ with $0 \neq r(1)\alpha \in M$. But $r(1)\alpha = (r)\alpha = (1)r\alpha$; so $r\alpha \in V(M)$. Also, $V(M)$ is $\mathscr{T}$-dense in $U(M)$. For if $\alpha \in U(M)$, then $(M\colon(1)\alpha) \in \mathscr{L}$ and $(M\colon(1)\alpha)\alpha \subseteq V(M)$; so $(V(M)\colon \alpha) \in \mathscr{L}$. A canonical epimorphism $\gamma_M\colon U(M) \to E_{\mathscr{T}}(M)$ is given by $\alpha \mapsto (1)\alpha$, and $V(M)$ is just $M\gamma_M^{-1}$. Also, $\gamma_M \in \pi^{-1}(\mathscr{F})$; for if $G \in \mathscr{F}$ and $\sigma\colon G \to E_{\mathscr{T}}(M)$, then σ can be extended to $\tau\colon E_{\mathscr{T}}(G) \to E_{\mathscr{T}}(M)$ by $\mathscr{T}$-injectivity. Now define $\tau'\colon G \to U(M)$ by $(g)\tau'\colon q \mapsto (qg)\tau$. Then $\tau'\gamma_M\colon g \mapsto (1)g\tau' = (g)\tau = (g)\sigma$; so $\tau'\gamma_M = \sigma$.

PROPOSITION 3.1. *For any module M, $U(M) \in \mathscr{F}$ if and only if $U(M) \cong \mathscr{F}(E_{\mathscr{T}}(M))$.*

PROOF. The "if" part is trivial. Assume that $U(M) \in \mathscr{F}$. The epimorphism $\gamma_M: U(M) \to E_{\mathscr{T}}(M)$ belongs to $\pi^{-1}(\mathscr{F})$; so γ_M is an $\mathscr{F}$-precover. Let K be a $\mathscr{T}$-pure submodule of $U(M)$ contained in $\ker(\gamma_M)$. If $\Sigma q_i k_i \in Q_{\mathscr{T}}K$, then $I = \cap(R: q_i) \in \mathscr{L}$. This implies that $Q_{\mathscr{T}}K/K \in \mathscr{T}$, which contradicts the $\mathscr{T}$-purity of K in $U(M)$ unless $Q_{\mathscr{T}}K = K$.

Hence K is a $Q_{\mathscr{T}}$-submodule of $U(M)$. If $K \neq 0$, then there is an $\alpha \in K$ and a $q \in Q_{\mathscr{T}}$ with $(q)\alpha \neq 0$. But then $(1)q\alpha = (q)\alpha \neq 0$; so $q\alpha \notin \ker(\gamma_M)$. Consequently $q\alpha \notin K$, which contradicts the fact that K is a $Q_{\mathscr{T}}$-module. Therefore $K = 0$, and hence γ_M is an $\mathscr{F}$-cover.

THEOREM 3.2. *Let S be an overring of a ring R such that the inclusion $R \to S$ is a left localization. Then there is a one-to-one correspondence between*

(1) *The class of all faithful hereditary torsion theories $(\mathscr{T}, \mathscr{F})$ on* R-mod *with $Q_{\mathscr{T}} = S$, and*

(2) *The class of all faithful hereditary torsion theories $(\mathscr{T}^*, \mathscr{F}^*)$ on* S-mod *with $Q_{\mathscr{T}^*} = S$.*

PROOF. Let $Q_{\mathscr{T}} = S$, and set $\mathscr{T}^* = \{{}_SN \mid {}_RN \in \mathscr{T}\}$. and $\mathscr{F}^* = \{{}_SN \mid {}_RN \in \mathscr{F}\}$. Then $\mathscr{T}^*$ is clearly closed under taking S-submodules, direct sums, S-homomorphic images, and group extensions. Also, $\mathscr{F}^*$ is closed under taking S-submodules, direct products, and group extensions. Furthermore, $S \in \mathscr{F}^*$.

If $T \in \mathscr{T}^*$ and $F \in \mathscr{F}^*$, then $\mathrm{Hom}_S(T, F) = 0$. Conversely, we claim that $\mathscr{T}(M)$ is an S-submodule of any ${}_SM$. Indeed, if $m \in \mathscr{T}(M)$ and $s \in S$, then there is an $I \in \mathscr{L}$ with $Is \subseteq R$. For each $a \in I$, $asm \in \mathscr{T}(M)$, and so there is an $H \in \mathscr{L}$ with $Hasm = 0$. This implies that $(0: sm) \in \mathscr{L}$, and hence $sm \in \mathscr{T}(M)$. Therefore, if $M \notin \mathscr{T}^*$, there is a nonzero S-homomorphism $M \to M/\mathscr{T}(M) \in \mathscr{F}^*$. Thus $\mathscr{T}^* = \{{}_ST \mid \mathrm{Hom}_S(T, F) = 0 \text{ for all } F \in \mathscr{F}^*\}$; so $(\mathscr{T}^*, \mathscr{F}^*)$ is a faithful hereditary torsion theory on S-mod with $Q_{\mathscr{T}^*} = S$.

Now assume that $(\mathscr{T}_1, \mathscr{F}_1)$ and $(\mathscr{T}_2, \mathscr{F}_2)$ are two faithful hereditary torsion theories on R-mod with $(\mathscr{T}_1^*, \mathscr{F}_1^*) = (\mathscr{T}_2^*, \mathscr{F}_2^*)$. Since $\mathscr{F}_1$ is closed under group extensions and submodules, $F \in \mathscr{F}_1$ if and only if $SF \in \mathscr{F}_1^*$. Similarly, $F \in \mathscr{F}_2$ if and only if $SF \in \mathscr{F}_2^*$. Since $\mathscr{F}_1^* = \mathscr{F}_2^*$, it follows that $\mathscr{F}_1 = \mathscr{F}_2$, and hence $(\mathscr{T}_1, \mathscr{F}_1) = (\mathscr{T}_2, \mathscr{F}_2)$. Thus the correspondence $(\mathscr{T}, \mathscr{F}) \mapsto (\mathscr{T}^*, \mathscr{F}^*)$ is monic.

Conversely, assume that $(\mathscr{T}^*, \mathscr{F}^*)$ is a faithful hereditary torsion theory on S-mod with $Q_{\mathscr{T}^*} = S$, and let $\mathscr{T} = \{{}_RM \mid S \otimes_R M \in \mathscr{T}^*\}$. Since S_R is flat, $\mathscr{T}$ is closed under taking R-modules, direct sums, R-homomorphic images, and group extensions; so $\mathscr{T}$ determines a hereditary torsion theory $(\mathscr{T},\mathscr{F})$. Since $R \to S$ is an epimorphism of rings, for any left S-module N, $S \otimes_R N \cong N$; so $\mathscr{T}^* = \{{}_S N \mid N \in \mathscr{T}\}$. Finally, since $\mathscr{T}^*(S) = 0$ implies $\mathscr{T}(S) = 0$, then $\mathscr{T}(R) = 0$; hence $(\mathscr{T},\mathscr{F})$ is faithful. Clearly $Q_{\mathscr{T}} = S$. Therefore the correspondence $(\mathscr{T},\mathscr{F}) \mapsto (\mathscr{T}^*,\mathscr{F}^*)$ is epic.

PROPOSITION 3.3. *If $R \to Q_{\mathscr{T}}$ is a left localization and if $M \in \mathscr{F}^*$, then M is $\mathscr{T}$-injective if and only if M is $\mathscr{T}^*$-injective.* (Notation is as in Theorem 3.2).

PROOF. Let $I \in \mathscr{L}$ and let $\alpha \in \mathrm{Hom}_R(I, M)$. Since $R \to Q_{\mathscr{T}}$ is a left localization it follows from Proposition 1.1 that $Q_{\mathscr{T}} I \cong Q_{\mathscr{T}} \otimes_R I$ and $Q_{\mathscr{T}} \otimes_R M \cong Q_{\mathscr{T}} M = M$. Since $Q_{\mathscr{T}}$ is flat, we obtain a $Q_{\mathscr{T}}$-homomorphism $\alpha' : Q_{\mathscr{T}} I \to M$ defined by α': $\Sigma q_i a_i \mapsto \Sigma q_i(a_i\alpha)$ which extends α. If M is $\mathscr{T}^*$-injective, there is a $Q_{\mathscr{T}}$-homomorphism $\beta' : Q_{\mathscr{T}} \to M$ extending α'. Since the restriction β of β' to R extends α, M is $\mathscr{T}$-injective.

Conversely, if M is $\mathscr{T}$-injective and if γ: $I \to M$ is a $Q_{\mathscr{T}}$-homomorphism, then there is an R-homomorphism $\delta : Q_{\mathscr{T}} \to M$ extending γ. For $q_1, q_2 \in Q_{\mathscr{T}}$ and $r \in (R : q_1)$, we have $r((q_1q_2)\delta - q_1(q_2)\delta) = r(q_1q_2)\delta - (rq_1)(q_2\delta) = (rq_1q_2)\delta - (rq_1q_2)\delta = 0$.

Since $M \in \mathscr{F}$, it follows that $q_1(q_2)\delta = (q_1q_2)\delta$, which proves that δ is a $Q_{\mathscr{T}}$-homomorphism. Therefore M is $\mathscr{T}^*$-injective as a $Q_{\mathscr{T}}$-module.

THEOREM 3.4. *Let $R \to Q = Q_{\mathscr{T}}$ be a left localization, and let $\mathscr{T}$ and $\mathscr{T}^*$ correspond as in Theorem 3.2. Then the following conditions are equivalent:*

(1) *$(\mathscr{T},\mathscr{F})$ is universally covering.*

(2) *$(\mathscr{T}^*,\mathscr{F}^*)$ is universally covering.*

PROOF. (1) $\Rightarrow$ (2): Let N be a left Q-module, and let μ: $F \to N$ be an $\mathscr{F}$-cover. Then μ induces a Q-homomorphism $\mu' : Q \otimes_R F \to N$ defined by $\mu' : \Sigma(q_i \otimes x_i) \mapsto \Sigma q_i(x_i\mu)$. By Proposition 1.1, $Q \otimes_R F \in \mathscr{F}$ and indeed is canonically isomorphic to QF. Thus we have a Q-epimorphism $\mu'' : QF \to N$ given by $\mu'' : \Sigma q_i x_i \mapsto \Sigma q_i(x_i\mu)$. Also $QF \subseteq E_{\mathscr{T}}(F)$, and so $QF \in \mathscr{F}^*$.

Now let $G \in \mathscr{F}^*$, and let $\sigma \in \mathrm{Hom}_Q(G, N)$. Then there is a $\tau \in \mathrm{Hom}_R(G, F)$ with $\sigma = \tau\mu$, and τ induces a Q-homomorphism $G \to QF$. Indeed, if $g \in G$ and $q \in Q$, then there is an $I \in \mathscr{L}$ with $Iq \subseteq R$; hence $(aqg)\tau = aq(g\tau)$ for all $a \in I$. Set $x = (qg)\tau - q(g\tau)$. Then $Ix = 0$; so $x \in \mathscr{T}(QF) = 0$, which implies that $(qg)\tau = q(g\tau)$. Therefore $\sigma = \tau\mu''$ as Q-homomorphisms, and thus $QF \in \pi^{-1}(\mathscr{F}^*)$.

Now suppose that K is a $\mathscr{T}^*$-pure submodule of QF contained in $\ker(\mu'')$. Then we have an R-monomorphism $F/[F \cap K] \to QF/K$, and hence $F/[F \cap K] \in \mathscr{F}$. But $F \cap K \subseteq \ker(\mu)$; so $F \cap K = 0$. Since F is large in QF, $K = 0$. Therefore μ'' is an $\mathscr{F}^*$-cover.

(2) $\Rightarrow$ (1): Let $\delta: V(M) \to M$ be the restriction of $\gamma_M: U(M) \to E_{\mathscr{T}}(M)$ to $V(M)$. Since $\gamma_M \in \pi^{-1}(\mathscr{F})$, then $\delta \in \pi^{-1}(\mathscr{F})$. By (2), we have an $\mathscr{F}^*$-cover $\mu': {}_QF' \to {}_QU(M)$. Let F be the inverse image of $V(M)$ under μ', and let $\mu: {}_RF \to {}_RV(M)$ be the restriction of μ' to F. Then $F \in \mathscr{F}$ and μ is an epimorphism. The composite map $\mu\delta: F \to M$ is also an epimorphism. If $G \in \mathscr{F}$ and $\sigma: G \to M$, then there is a homomorphism $\psi: G \to V(M)$ with $\sigma = \psi\delta$. Furthermore, ψ induces $\psi': {}_Q(Q \otimes_R G) \to {}_QU(M)$ defined by $\psi': \Sigma(q_i \otimes g_i) \mapsto \Sigma q_i(g_i\psi)$. By Proposition 1.1, $Q \otimes_R G \in \mathscr{F}^*$, and hence there is a Q-homomorphism $\phi: Q \otimes_R G \to F'$ with $\psi' = \phi\mu'$. Also, G is embedded in $Q \otimes_R G$, and $G\phi\mu' = G\psi' = G\psi \subseteq V(M)$; so $G\phi \subseteq F$, and $\phi\mu\delta = \psi\delta = \sigma$. Hence $\mu\delta \in \pi^{-1}(\mathscr{F})$.

Now let K be a $\mathscr{T}$-pure submodule of F contained in $\ker(\mu\delta)$. We first note that $F\mu \cap (QK)\mu' = (F \cap QK)\mu$. Indeed, if $y = x\mu = (\Sigma q_ik_i)\mu' \in F\mu \cap (QK)\mu'$; then $0 = (x - \Sigma q_ik_i)\mu'$; so $x - \Sigma q_ik_i \in \ker(\mu') \subseteq F$. Therefore $\Sigma q_ik_i \in F \cap QK$, which implies that $y \in (F \cap QK)\mu$. The reverse inclusion is trivial.

Clearly $F \cap QK \supseteq K$. If $\Sigma q_ik_i \in F \cap QK$, then $I = \cap(R: q_i) \in \mathscr{L}$ and $\Sigma Iq_ik_i \subseteq K$. Thus $(F \cap QK)/K \in \mathscr{T}$. Since K is $\mathscr{T}$-pure in F, it follows that $K = F \cap QK$.

Now $V(M) \cap (QK)\mu' = F\mu \cap (QK)\mu' = (F \cap QK)\mu = K\mu \subseteq \ker(\delta)$. Therefore, if $\alpha \in V(M) \cap (QK)\mu'$ and $q \in Q$ with $q\alpha \in V(M)$, then $(q)\alpha = (1)q\alpha = (q\alpha)\delta = 0$ Since M is large in $E_{\mathscr{T}}(M)$, this implies that $Q\alpha = 0$, and so $\alpha = 0$. Thus $0 = K\mu$, and hence $K \subseteq \ker(\mu)$. Also, $(QK)\mu' = Q(K\mu') = 0$; so $QK \subseteq \ker(\mu')$.

Let $W/QK = \mathscr{T}(QF/QK)$, and let $w = \Sigma q_ix_i \in W$. Then $I = \cap(R: q_i)$ belongs to $\mathscr{L}$ and $aw \in W \cap F$ for every $a \in I$. Then there is an $H \in \mathscr{L}$ with $Haw \subseteq QK \cap F = K$, which implies that $aw + K \in \mathscr{T}(F/K) = 0$. Therefore $aw \in K$. Since this is true for any $a \in I$, $Iw \subseteq K$, and hence $w + K \in \mathscr{T}(QF/K)$.

Since $V(M)$ is large in $U(M)$, $V(M)$ is also large in $QV(M)$. Since inverse images of large submodules are large, F/K is large in QF/K. Since $F/K \in \mathscr{F}$, this forces $QF/K \in \mathscr{F}$; so by the above paragraph, $QF/QK \in \mathscr{F}$. Again, $QV(M)$ is large in $U(M)$; so QF/QK is large in F'/QK, which implies that $F'/QK \in \mathscr{F}$. Hence $F'/QK \in \mathscr{F}^*$. Since $K \subseteq QK \subseteq \ker(\mu')$ and μ' is an $\mathscr{F}^*$-cover, then $K = 0$.

As a consequence of Theorem 3.4, we obtain the following major result of [3].

COROLLARY 3.5. *The usual torsion theory for modules over an integral domain is universally covering.*

PROOF. If R is an integral domain and $(\mathscr{T}, \mathscr{F})$ is the usual torsion theory, then $Q_{\mathscr{T}} \in \mathscr{F}$ is a field. Therefore every $Q_{\mathscr{T}}$-module is a direct sum of copies of $Q_{\mathscr{T}}$ and hence is in $\mathscr{F}$. Thus every $Q_{\mathscr{T}}$-module is in $\mathscr{F}^*$, i.e., $(\mathscr{T}^*, \mathscr{F}^*)$ is (trivially) universally covering. Hence the result follows from Theorem 3.4.

As in [10], $(\mathscr{T}, \mathscr{F})$ is called *perfect* if every left $Q_{\mathscr{T}}$-module is in $\mathscr{F}$ when considered as an R-module in the natural way. [6, Th. 4.3] and [10, Th. 13.1] give several conditions equivalent to $(\mathscr{T}, \mathscr{F})$ being perfect.

We now obtain the following result of Banaschewski [1, p. 66].

COROLLARY 3.6. *A perfect torsion theory is universally covering.*

PROOF. If $(\mathscr{T}, \mathscr{F})$ is perfect, then every $Q_{\mathscr{T}}$-module is in $\mathscr{F}^*$; so $(\mathscr{T}^*, \mathscr{F}^*)$ is (trivially) universally covering. By Theorem 3.4, $(\mathscr{T}, \mathscr{F})$ is also universally covering.

Indeed, $\mathscr{F}(M) = V(M)$ whenever $(\mathscr{T}, \mathscr{F})$ is perfect.

COROLLARY 3.7. *If $R \to Q_{\mathscr{T}}$ is a left localization and $Q_{\mathscr{T}}$ satisfies the ascending chain condition on $\mathscr{T}^*$-injective left ideals, then $(\mathscr{T}, \mathscr{F})$ is universally covering. In particular, if $Q_{\mathscr{T}}$ is noetherian, then $(\mathscr{T}, \mathscr{F})$ is universally covering.*

PROOF. This follows by combining Theorem 3.4 with Theorems 1.2 and 2.4 of [11].

EXAMPLE 3.8. Let $R = K[X, Y]$ be a commutative polynomial ring over a field K. The maximal ideal $H = (X, Y)$ determines an idempotent filter $\mathscr{L}_H = \{{}_RI \subseteq R \mid I \supseteq H^n \text{ for some } n\}$. Let $(\mathscr{T}_H, \mathscr{F}_H)$ be the hereditary torsion theory associated with $\mathscr{L}_H$. By [10, p. 40, Exercise 3], $Q_{\mathscr{T}_H} = R$ and $(\mathscr{T}_H, \mathscr{F}_H)$ is not perfect. Since $Q_{\mathscr{T}_H} = R$ is noetherian, Corollary 3.7 implies that $(\mathscr{T}_H, \mathscr{F}_H)$ is universally covering; however, since $(\mathscr{T}_H, \mathscr{F}_H)$ is not perfect, Corollary 3.6 is not applicable to this situation.

LEMMA 3.9. *The following statements are equivalent.*

(1) *$Q_{\mathscr{T}}$ has the ascending chain condition on $\mathscr{T}$-pure left $Q_{\mathscr{T}}$-ideals.*

(2) *$Q_{\mathscr{T}}$ has the ascending chain condition on $\mathscr{T}$-injective left $Q_{\mathscr{T}}$-ideals.*

(3) *R has the ascending chain condition on $\mathscr{T}$-pure left ideals.*

PROOF. (1) $\Rightarrow$ (2) follows from [6, Prop. 3.3].

(2) $\Rightarrow$ (3): If $K_1 \subset K_2 \subset \cdots$ is an ascending chain of $\mathscr{T}$-pure left ideals of R, then $E_{\mathscr{T}}(K_1) \subset E_{\mathscr{T}}(K_2) \subset \cdots$ is an ascending chain of $\mathscr{T}$-injective left $Q_{\mathscr{T}}$-ideals.

Thus for some m, $E_{\mathscr{T}}(K_m) = E_{\mathscr{T}}(K_{m+i})$ for $i \geqq 1$. This implies that $K_m = K_{m+i}$ for $i \geqq 1$.

(3) $\Rightarrow$ (1): If $K_1 \subset K_2 \subset \cdots$ is an ascending chain of $\mathscr{T}$-pure left $Q_{\mathscr{T}}$-ideals, set $L_i = R \cap K_i$ for each i. Since $R/L_i = R/(R \cap K_i) \cong (R + K_i)/K_i \subseteq Q_{\mathscr{T}}/K_i \in \mathscr{F}$, it follows from (3) that $L_m = L_{m+1} = \cdots$ for some integer m. But $K_i/L_i = K_i/(R \cap K_i) \cong (K_i + R)/R \subseteq Q_{\mathscr{T}}/R \in \mathscr{T}$. Consequently, $K_i = E_{\mathscr{T}}(L_i)$, and hence $K_m = K_{m+j}$ for $j \geqq 1$.

A special case of [11, Theorem 2.4] now follows from Corollary 3.7.

COROLLARY 3.10. *If $R \to Q_{\mathscr{T}}$ is a left localization and if any direct sum of injective modules in $\mathscr{F}$ is injective, then $(\mathscr{T}, \mathscr{F})$ is universally covering.*

PROOF. The direct sum hypothesis and [11, Th. 1.2] imply that R has the ascending chain condition on $\mathscr{T}$-pure left ideals. But this implies that condition (3) of Lemma 3.9 holds. From Lemma 3.9(2), Proposition 3.3, and Corollary 3.7, it then follows that $(\mathscr{T}, \mathscr{F})$ is universally covering.

The hereditary torsion theories on R-mod are partially ordered by $(\mathscr{T}', \mathscr{F}') \leqq (\mathscr{T}, \mathscr{F})$ if and only if $\mathscr{T}' \subseteq \mathscr{T}$.

PROPOSITION 3.11. *Let $R \to Q_{\mathscr{T}}$ be a proper left localization. Then there exists a non-trivial, universally-covering torsion theory $(\mathscr{T}', \mathscr{F}') \leqq (\mathscr{T}, \mathscr{F})$ with $Q_{\mathscr{T}'} = Q_{\mathscr{T}}$.*

PROOF. By [10, Th. 13.10], the set $\{I \mid Q_{\mathscr{T}}I = Q_{\mathscr{T}}\}$ is an idempotent filter of left ideals which determines a torsion theory $(\mathscr{T}', \mathscr{F}')$ such that $Q_{\mathscr{T}'} = Q_{\mathscr{T}}$. If $\zeta_M: M \to Q_{\mathscr{T}} \otimes_R M: m \mapsto 1 \otimes m$, then $\mathscr{T}'(M) = \ker(\zeta_M)$ by [10, Th. 13.1]; so $\mathscr{F}' = \{M \mid \ker(\zeta_M) = 0\} \supseteq \mathscr{F}$. Hence $\mathscr{T}' \subseteq \mathscr{T}$, and $(\mathscr{T}', \mathscr{F}') \leqq (\mathscr{T}, \mathscr{F})$. Since $Q_{\mathscr{T}}/R$ is a nonzero member of $\mathscr{T}'$ and $R \in \mathscr{F}'$, then $(\mathscr{T}', \mathscr{F}')$ is non-trivial. By [6, Th. 4.3] and Corollary 3.6, every module has an $\mathscr{F}'$-cover.

A theory $(\mathscr{T}, \mathscr{F})$ is *primary* if it has no non-trivial theories smaller than itself. The primary theories are precisely those determined by simple left R-modules [2].

COROLLARY 3.12. *If $(\mathscr{T}, \mathscr{F})$ is primary and $R \to Q_{\mathscr{T}}$ is a proper left localization, then $(\mathscr{T}, \mathscr{F})$ is universally covering.*

THEOREM 3.13. *If the functor $Q_{\mathscr{T}}(\_)$ is exact, then the following conditions are equivalent:*

(1) *$\mathscr{E}$ is closed under taking direct sums.*

(2) *$(\mathscr{T}, \mathscr{F})$ is perfect.*

(3) *$(\mathscr{T}, \mathscr{F})$ is universally covering.*

(4) *$\mathscr{L}$ contains a cofinal subset of finitely-generated left ideals*

PROOF. In view of [6, Th. 4.3], [10, Th. 13.1], and Corollary 3.6, we only need to prove (3) $\Rightarrow$ (2). Let M be a $Q_{\mathscr{T}}$-module, and let $m \in \mathscr{T}({}_R M)$. Then $Q_{\mathscr{T}} m \cong Q_{\mathscr{T}}/I \in \mathscr{T}$. If $Q_{\mathscr{T}}/I \neq 0$, let $\mu \colon \mathscr{F}(Q_{\mathscr{T}}/I) \to Q_{\mathscr{T}}/I$ be an $\mathscr{F}$-cover by (3). Then there exists a $\alpha \in \operatorname{Hom}_R(Q_{\mathscr{T}}, \mathscr{F}(Q_{\mathscr{T}}/I))$ such that $\alpha\mu = \nu$, where $\nu \colon Q_{\mathscr{T}} \to Q_{\mathscr{T}}/I$ is canonical. Then $I\alpha \neq 0$; for otherwise α induces a nonzero homomorphism from $Q_{\mathscr{T}}/I$ into $\mathscr{F}(Q_{\mathscr{T}}/I)$. Hence there exists $x \in I$ such that $0 \neq Q_{\mathscr{T}} x\alpha \subseteq I\alpha \subseteq \ker(\mu)$. But $Q_{\mathscr{T}} x\alpha \in \mathscr{E}$ by [6, Th. 4.5], which contradicts the definition of μ.

If $\mathscr{L}$ has a cofinal subset of projective right ideals, then $Q_{\mathscr{T}}(\_)$ is right exact [6, Th. 4.5]; thus we have the following two corollaries of Theorem 3.13.

COROLLARY 3.14. *If R is left hereditary then $(\mathscr{T}, \mathscr{F})$ is universally covering if and only if it is perfect.*

COROLLARY 3.15. *If R is left semi-hereditary and if $\mathscr{L}$ has a cofinal subset of finitely generated left ideals, then $(\mathscr{T}, \mathscr{F})$ is universally covering.*

4. Products of covers

We need an elementary, but useful, result on the ordering. For the next result only, we drop our assumption that all torsion theories mentioned are faithful.

PROPOSITION 4.1. *Let $(\mathscr{T}, \mathscr{F})$ be a faithful hereditary torsion theory, and let $(\mathscr{T}', \mathscr{F}')$ be a hereditary torsion theory on R-mod with associated filters, $\mathscr{L}$ and $\mathscr{L}'$, respectively. Then the following conditions are equivalent:*

(1) $(\mathscr{T}, \mathscr{F}) \leq (\mathscr{T}', \mathscr{F}')$.

(2) $\mathscr{T} \cap \mathscr{F}' = 0$.

(3) *$R/I \notin \mathscr{F}'$ for all proper left ideals $I \in \mathscr{L}$.*

(4) *$Q_{\mathscr{T}}/K \notin \mathscr{F}'$ for all $K \subsetneq Q_{\mathscr{T}}$ with $R \cap K \in \mathscr{L}$.*

(5) *${}_R M$ is $\mathscr{T}$-injective if and only if $\operatorname{Ext}_R(R/H, M) = 0$ for every left ideal $H \in \mathscr{L} \cap \mathscr{L}'$.*

PROOF. (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (5) can be easily verified by the reader.

(5) $\Rightarrow$ (4): Let $K \subsetneq Q_{\mathscr{T}}$ with $R \cap K \in \mathscr{L}$, and assume that $Q_{\mathscr{T}}/K \in \mathscr{F}'$. We first claim that $\operatorname{Ext}_R(R/H, K) = 0$ for any left ideal $H \in \mathscr{L} \cap \mathscr{L}'$. Indeed, if $H \in \mathscr{L} \cap \mathscr{L}'$ and if $\alpha \in \operatorname{Hom}_R(H, K)$ then by the $\mathscr{T}$-injectivity of $Q_{\mathscr{T}}$ there is an R-homomorphism $\beta \colon R \to Q_{\mathscr{T}}$ such that $\beta|_H = \alpha$.

Then β induces a homomorphism $\beta' \colon R/H \to Q_{\mathscr{T}}/K$. Since $\operatorname{im} \beta' \in \mathscr{T}' \cap \mathscr{F}' = 0$, then $R\beta \subseteq K$. Therefore $\operatorname{Hom}_R(R, K) \to \operatorname{Hom}_R(H, K) \to 0$ is exact; so it follows that $\operatorname{Ext}_R(R/H, K) = 0$.

By (5), this implies that K is in $\mathscr{E}$. In particular, $[K+R]/K \subseteq Q_{\mathscr{T}}/K \in \mathscr{F}$. On the other hand, $[K+R]/K \cong R/[K \cap R] \in \mathscr{T}$. Since $\mathscr{T} \cap \mathscr{F} = 0$, we must have $K + R = K$, and so $R \subseteq K$. Thus $Q_{\mathscr{T}}/K$ is a nonzero homomorphic image of $Q_{\mathscr{T}}/R$; so $Q_{\mathscr{T}}/K \in \mathscr{T}$, which contradicts the $\mathscr{T}$-purity of K in $Q_{\mathscr{T}}$.

(4) ⇒ (3): Let $I \in \mathscr{L}$ with $I \neq R$. Since $Q_{\mathscr{T}}/I \notin \mathscr{F}'$ by (4), then $W/I = \mathscr{T}'(Q_{\mathscr{T}}/I) \neq 0$. Then $I \subseteq R \cap W$; so $R \cap W \in \mathscr{L}$. But $Q_{\mathscr{T}}/W \in \mathscr{F}'$, and hence by (4) we have $W = Q_{\mathscr{T}}$. Thus $Q_{\mathscr{T}}/I \in \mathscr{T}'$, which surely implies (3).

Let $\mathscr{S}$ be the smallest torsion class containing all the simple modules, and let $(\mathscr{S}, \mathscr{C})$ be a torsion theory. Then $\mathscr{S}$ is hereditary [2].

THEOREM 4.2. *For a universally-covering theory* $(\mathscr{T}, \mathscr{F})$, *the following conditions are equivalent*:

(1) *The product of* $\mathscr{T}$*-neat homomorphisms is* $\mathscr{T}$*-neat.*

(2) *The product of* $\mathscr{F}$*-covers is* $\mathscr{T}$*-neat.*

(3) $(\mathscr{T}, \mathscr{F}) \leqq (\mathscr{S}, \mathscr{C})$.

PROOF. (1) ⇒ (2) since every $\mathscr{F}$-cover is $\mathscr{T}$-neat.

(2) ⇒ (3): By Proposition 4.1, it suffices to show that, for any ${}_RK \subsetneqq Q_{\mathscr{T}}$ with $R \cap K \in \mathscr{L}$, $\mathrm{soc}(Q_{\mathscr{T}}/K) \neq 0$. Pick such a module K, and let Ω index the family of R-submodules V_i of $Q_{\mathscr{T}}$ properly containing K. For each $i \in \Omega$, let $M_i = V_i/K$, and let $\mu_i : F_i \to M_i$ be an $\mathscr{F}$-cover. Set $M = \Pi M_i$, $F = \Pi F_i$, and $\mu = \Pi \mu_i$. For each $i \in \Omega$ and each canonical epimorphism $\mu_i : F_i \to V_i/K = M_i$ there exists a homomorphism $\sigma_i : V_i \to F_i$ such that $\sigma_i \mu_i = \nu_i$. Define $\sigma \in \mathrm{Hom}_R(K, F)$ by $\sigma : k \mapsto \langle k\sigma_i \rangle$. Then $K\sigma \subseteq \ker(\mu)$, and the 0-homomorphism $Q_{\mathscr{T}} \to M$ extends $\sigma\mu$. By (2), μ is $\mathscr{T}$-neat; so there is a submodule W of $Q_{\mathscr{T}}$ properly containing K and a $\tau \in \mathrm{Hom}_R(W, F)$ extending σ. It is now sufficient to show that W/K is completely reducible.

Let $0 \neq V/K \subseteq W/K$. For some $j \in \Omega$, $V/K = M_j$. Let $\pi_j : F \to F_j$ be the canonical projection. If $\rho = \sigma_j - (\tau|_V)\pi_j$, then $K\rho = 0$. Therefore ρ induces a homomorphism $\rho' : V/K \to F_j$. Since $R/[R \cap K]$ and $Q_{\mathscr{T}}/R$ both belong to $\mathscr{T}$, then $Q_{\mathscr{T}}/[R \cap K] \in \mathscr{T}$. Therefore $V/[R \cap K] \in \mathscr{T}$, and so $V/K \in \mathscr{T}$. Since $F_j \in \mathscr{F}$, this means that $\rho' = 0$, and thus $\rho = 0$. Therefore $(\tau|_V)\pi_j\mu_j = \sigma_j\mu_j = \nu_j$; hence $\tau\pi_j\mu_j : W \to V/K$ induces a homomorphism $W/K \to V/K$, the restriction of which to V/K is the identity. Hence V/K is a direct summand of W/K. Since this is true for all V, W/K is completely reducible.

(3) ⇒ (1): Let $\{\phi_i : N_i \to M_i \mid i \in \Omega\}$ be any family of $\mathscr{T}$-neat homomorphisms.

Set $M = \Pi M_i$, $N = \Pi N_i$, and $\phi = \Pi\phi_i$. For each $i \in \Omega$, let $\pi_i : M \to M_i$ and $\pi_i' : N \to N_i$ be the canonical projections. Let U be a $\mathscr{T}$-dense submodule of an arbitrary left R-module V, and let $\sigma \in \mathrm{Hom}_R(U, N)$. Assume that there is a submodule H' of V properly containing U and a homomorphism $\beta' : H' \to M$ extending $\sigma\phi$. Since $V/U \in \mathscr{T}$, then H'/U has a simple submodule H/U by (3). Let β be the restriction of β' to H. Then β properly extends $\sigma\phi$, and, for each $i \in \Omega$, $\beta\pi_i$ extends $\sigma\phi\pi_i = \sigma\pi_i'\phi_i$. Since each ϕ_i is $\mathscr{T}$-neat, there are proper extensions ψ_i of $\sigma\pi_i'$ in H. Since H/U is simple, the domains of the ψ_i must all equal H. Define $\psi \in \mathrm{Hom}_R(H, N)$ by $\psi : h \mapsto \langle h\psi_i \rangle$. Then ψ properly extends σ, which proves that ϕ is $\mathscr{T}$-neat.

The following result generalizes Theorem 4.1 of [4].

COROLLARY 4.3. *Let $(\mathscr{T}, \mathscr{F})$ be universally-covering, and assume that $Q_{\mathscr{T}}(\_)$ is exact. Then the product of $\mathscr{F}$-covers is an $\mathscr{F}$-cover if and only if $(\mathscr{T}, \mathscr{F}) \leqq (\mathscr{S}, \mathscr{C})$.*

PROOF. "Only if": This follows directly from Theorem 4.2 and the fact that every $\mathscr{F}$-cover is $\mathscr{T}$-neat.

"If": Let $\{\mu_i : F_i \to M_i \mid i \in \Omega\}$ be a family of $\mathscr{F}$-covers. Let $F = \Pi F_i$ $M = \Pi M_i$, and $\mu = \Pi\mu_i$. For each $i \in \Omega$, let $\pi_i : F \to F_i$ be the canonical projection. Then μ clearly satisfies the definition of an $\mathscr{F}$-precover, and μ is $\mathscr{T}$-neat by Theorem 4.2. If W is a $\mathscr{T}$-pure submodule of F contained in $\ker(\mu)$, then W is in $\mathscr{E}$ by Proposition 2.10. Since $Q_{\mathscr{T}}(\_)$ is exact, $W\pi_i$ is also in in $\mathscr{E}$ for each $i \in \Omega$ [6, Theorem 4.5]. Since each μ_i is an $\mathscr{F}$-cover and $W\pi_i \subseteq \ker(\mu_i)$, then $W\pi_i = 0$ for each $i \in \Omega$. Therefore $W = 0$. This proves that μ is an $\mathscr{F}$-cover.

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